THE FOUR COLOUR CONJECTURE

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It was in 1852 that Francis Guthrie, then a student in London wrote to his brother Frederick Guthrie that every map could always be coloured with not more than four colours. Augustus DeMorgan, then a prominent mathematician and teacher of the Guthrie brothers, got interested in the proposition and made the points: (i) that there are maps that do need four colours for the colouring scheme and (ii) that it is not possible for five countries to be such that each country is adjacent to each of the remaining four countries. This meant that for a colouring scheme of a map (drawn on a plane sheet) four colours are necessary but five colours are sufficient. To enable the proposition to be mathematically precise, we must invoke strict definitions of 'colour', 'map', 'properly', 'neighbouring countries', etc. Do we ever need more than four colours for a map drawn on a plane sheet, if the countries involved are connected regions and no country surrounds another country? Employing brute force to check the truth of the proposition either by manual effort or even using a computer is not a sound proposition.

The four colour conjecture is to conclude the truth or falsity of the proposition that every map can be coloured using a maximum of four colours. This conjecture is a rare instance of an easily understandable proposition and has interested the amateur as much as the professional mathematician.

The problem was proposed in 1878 by Arthur Cayley for investigation at a meeting of the London Mathematical Society and within a year, a solution proving the truth of the conjecture, was published by Arthur Bray Kempe, a London barrister who was also a member of the London Mathematical Society. There was considerable ingenuity in Kempe's approach, but there was a serious flaw in the argument and the conjecture remained unresolved in 1890 when the flaw was noticed by Percey John Heawood. Heawood examined the colouring problem in a more general setting and employing an elegant method, he was able to obtain an upper bound for the number of colours needed for maps on complicated surfaces like the petzel containing twists and hurdles. However, this method did not succeed in the resolution of the original four colour conjecture for maps on plane sheets or spheres.

To proceed strictly mathematically in the resolution of the conjecture, some notions are essential. A map is normal if none of its countries encloses other countries and not more than three countries meet at any point. To prove the four colour conjecture it suffices to prove that a normal five-chromatic map (i.e. a map needing five colours) is not possible. Parts of Kempe's paper contained ideas and concepts that seemed quite relevant for the testing of the four colour conjecture. Two ideas in the paper concerned "unavoidability" and "reducibility" and these played significant role in the eventual resolution of the conjecture.

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Every normal map on a plane sheet has at least one country that has (i) exactly two or (ii) exactly three or (iii) exactly four or (iv) exactly five neighbours. No normal map can have every country in it having six or more neighbours. Every normal map must contain at least one of the four configurations. The set of configurations consisting of a country with 2 or a country with 3 or a country with 4 or a country with 5 neighbours is called an "unavoidable" set.

A configuration is "reducible" if there is a way of showing that it cannot appear in a minimal five-chromatic map. Methods of proving the reducibility of configurations grew from Kempe's proof that a country with four neighbours cannot occur in a minimum five-chromatic map. He proved that if a five-chromatic map contained a country with four neighbours, there would be a five-chromatic map with a reduced number of countries.

The American mathematician G.D. Birkhoff (1884-1944) initiated some new techniques to resolve the four colour conjecture and using Kempe's idea, he was able to show that certain large configurations were 'reducible.' Philip Franklin came out with the assertion that a map hypothetically needing five colours (i.e. a five-chromatic map) must contain at least 22 countries. This number (called Birkhoff number of Franklin) was seem to rise to 36 by the further use of reducible configuration. This was the scenario regarding the four colour problem in 1950.

Heinrich Heesch began his work on the four colour conjecture in 1936 and was the first to publicise that the conjecture could be proved by finding an 'unavoidable' set of 'reducible' configurations. He conjectured (1950) that such a set could be found and that it would contain configurations of certain restricted sizes and that their number would be of the order of 10000. This was a watershed in the history of the four colour problem. Heesch and his student Karl Dürre wrote computer programs to prove the reducibility of configurations. The data generated by the program and further calculations using a stronger technique originally due to G.D. Birkhoff, proved helpful in proving the reducibility of configurations. Obstacles in the process of proving reducibility were got over by the use of the principle of duality. In the original map, the capital of each country is marked and when two countries are neighbours, their capitals are joined by a road that runs across their common border. Capitals (vertices) and roads (arch or edges) constitute the dual of the original map. The original map is a 'normal' one and the vertices of this map are such that there are three edges that join at the vertices. The vertices of the original map correspond to the faces in the dual map and hence the dual graph is triangulated. Testing the configurations for reducibility became a simplified task, thanks to the use of duality and the consequent triangulation of the map. Heesch's work on the theory of reducible configurations was extremely were developed and all the ideas on reducibility needed for proving the four colour conjecture were well understood during the decade 1960-1970. Attention on methods for finding unavoidable sets of configurations needed reinforcement and Heesch was able to introduce the "method of discharging", which turned out to be crucial in all later work on unavoidable sets. It became the central element in the proof of the four colour theorem. The stage was ready by 1972

for dialogue with a computer to obtain an unavoidable set of geographically good configurations.

The man-machine dialogue initiated in 1972 went on for months till there appeared a beacon of hope that there was indeed a feasible method of obtaining an unavoidable set of geographically good configurations. A hope had been held all along that the computer method tried would also provide a finite unavoidable set of geographically good configurations. To see this, the purely experimental process on the computer had to be kept aside and a total procedure was necessary to see that all cases are covered and that those that were not handled by the computer programs were as simple as they appeared. This was an extremely tough task and the entire investigation took a full year of study and testing. There was the necessity to prove abstract statements involving general definitions of terms and special cases had to be dealt with in every detail, and this often needed complicated mathematical analysis. The end result was a lengthy proof that a finite number of geographically good configurations was possible and that this constituted an unavoidable set. Precise bounds on the size(s) of the configurations in the unavoidable set were obtained, but these were much larger than desirable. This was the scenario in 1974. In sum, a chain of five mathematicians stretching from Kempe to Appel and Haken, and comprising stalwarts like G.D. Birkhoff, Philip Franklin, Heinrich Heesch, Karl Dürre, Walten Stromquist et al and using traditional, regular and mental mathematics, had finally reduced the problem to investigation of a finite number of instances. This number was so large that purely human effort at examining the repetitive questions involved was just ruled out and appeal to the use of the computer was thus a virtual necessity. However, the algorithm for computerising the question was indeed the contribution of mathematics and this stretched well beyond a full hundred years. It is not that the computer settled the truth of the four colour conjecture. Kenneth Appel and Wolfgang Haken who directed the final work by the computer have indeed made no such claim regarding the solution of the problems by the computer. Here are their own words: "The fundamental reason that the unavoidable set argument worked whereas other approaches to the Four-colour conjecture did not, is that all other approaches need somewhat stronger theoretical tools to make their methods apply. While these might be possible to create, there is no guarantee that they are actually possible, and if they are, there is no obvious way to go about finding them.

On the other hand, many mathematicians have believed that an unavoidable set of reducible configurations might exist, but that a smallest such was beyond the bounds of reasonable computation. This attitude appears justified when the problem is considered with respect to the tools available prior to 1960. After 1960, with the advent of faster computers, there were still strong reasons to believe that the computations would be infeasibly large, but there were certainly no theoretical difficulties to overcome other than the choice of a method for obtaining an unavoidable set. Thus by 1970 it became a problem of discovering whether efficient use of known techniques and technical (as opposed to theoretical) improvements would enable one to find an unavoidable set of reducible configurations.

Most mathematicians who were educated prior to the development of fast computers tend not to think of the computers as a routine tool to be used in conjunction with other older and more theoretical tools in advancing mathematical knowledge. Thus, they intuitively feel that if an argument contains parts that are not verifiable by hand calculations, it is on rather insecure ground. There is a tendency to feel that verification of computer results by independent computer programs is not as certain to be correct as independent hand checking of the proof of theorems proved in the standard way.

This point of view is reasonable for those theorems whose proofs are of moderate length and highly theoretical. When proofs are long and highly computational, it may be argued that even when hand checking is possible, the probability of human error is considerably higher than that of machine error. Moreover, if the computations are sufficiently routine, the validity of programs themselves is easier to verify than the correctness of hand computations.

In any event, even if the Four Colour Theorem turns out to have a simpler proof, mathematicians might well be advised to consider more carefully other problems that might have solutions of this new type, requiring computation or analysis of a type not possible for humans alone. There is every reason to believe that there are a large number of such problems. After all, the argument that almost all known proofs are reasonably short can be answered by the argument that if one only employs tools which will yield short proofs, that is all one is likely to get.

The example of the Four Colour Theorem may help to clarify the possibilities and the limitations of the methods of pure mathematics and those of computation. It may be that a problem cannot be solved by either of these alone but can be solved by a combination of the two methods."

The above is a balanced estimate of the role of the computer in the solution of the Four Colour problem.

From the time of Plato until the sixteenth century, mathematical methods were considered superior to experimental methods and this handicapped the development of certain branches of physics. After Galileo the importance of experimentation was recognized and a very fruitful development of physics was achieved by a combination of experimental and mathematical methods. May be, combination of computer use and mathematical arguments wherever feasible or desirable, is an index of development of knowledge in the direction of progress.

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